Differential Equation and Applications

EXERCISE 8.1 [PAGE 162]

Exercise 8.1 | Q 1.1 | Page 162

Determine the order and degree of the following differential equations.

$$\frac{d^2x}{dt^2} + \left(\frac{dx}{dt}\right)^2 + 8 = 0$$

Solution:

$$\frac{d^2x}{dt^2} + \left(\frac{dx}{dt}\right)^2 + 8 = 0$$

By definition of order and degree,

Order : 2 ; Degree : 1

Exercise 8.1 | Q 1.2 | Page 162

Determine the order and degree of the following differential equations.

$$\left(\frac{d^2y}{dx^2}\right)^2 + \left(\frac{dy}{dx}\right)^2 = a^x$$

Solution:

$$\left(\frac{d^2y}{dx^2}\right)^2 + \left(\frac{dy}{dx}\right)^2 = a^x$$

By definition of order and degree,

Order : 2 ; Degree : 2

Exercise 8.1 | Q 1.3 | Page 162

Determine the order and degree of the following differential equations.

$$\frac{d^4y}{dx^4} + \left[1 + \left(\frac{dy}{dx}\right)^2\right]^3 = 0$$

Solution:

$$\frac{d^4y}{dx^4} + \left[1 + \left(\frac{dy}{dx}\right)^2\right]^3 = 0$$

By definition of order and degree,

Order : 4 ; Degree : 1

Exercise 8.1 | Q 1.4 | Page 162

Determine the order and degree of the following differential equations.

$$(y''')^2 + 2(y'')^2 + 6y' + 7y = 0$$

Solution:

$$(y''')^2 + 2(y'')^2 + 6y' + 7y = 0$$

By definition of order and degree,

Order : 3 ; Degree : 2

Exercise 8.1 | Q 1.5 | Page 162

Determine the order and degree of the following differential equations.

$$\sqrt{1 + \frac{1}{\left(\frac{dy}{dx}\right)^2}} = \left(\frac{dy}{dx}\right)^{\frac{3}{2}}$$

Solution:

$$\sqrt{1+\frac{1}{\left(\frac{dy}{dx}\right)^2}}=\left(\frac{dy}{dx}\right)^{\frac{3}{2}}$$

Squaring on both sides, we get

$$1+\frac{1}{\left(\frac{dy}{dx}\right)^2}=\left(\frac{dy}{dx}\right)^3$$

$$\therefore \left(\frac{dy}{dx}\right)^2+1=\left(\frac{dy}{dx}\right)^5$$

By definition of order and degree,

Order : 1 ; Degree : 5

Exercise 8.1 | Q 1.6 | Page 162

Determine the order and degree of the following differential equations.

$$\frac{dy}{dx}=7\frac{d^2y}{dx^2}$$

Solution:

$$\frac{dy}{dx}=7\frac{d^2y}{dx^2}$$

By definition of order and degree,

Order : 2 ; Degree : 1

Exercise 8.1 | Q 1.7 | Page 162

Determine the order and degree of the following differential equations.

$$\left(\frac{d^3y}{dx^3}\right)^{\frac{1}{6}}=9$$

Solution:

$$\left(\frac{d^3y}{dx^3}\right)^{\frac{1}{6}} = 9$$

Taking sixth power on both sides, we get

$$\frac{d^3y}{dx^3} = 9^6$$

By definition of order and degree,

Order : 3 ; Degree : 1

Exercise 8.1 | Q 2.1 | Page 162

In each of the following examples, verify that the given function is a solution of the corresponding differential equation.

Solution	D.E.
xy = log y +k	y' (1-xy) =y2

Solution: xy = log y + k

Differentiating w.r.t. x, we get

$$x\frac{dy}{dx} + y(1) = \frac{1}{y}.\frac{dy}{dx}$$

$$\therefore xy\frac{dy}{dx} + y^2 = \frac{dy}{dx}$$

$$\therefore \frac{dy}{dx} - xy = \frac{dy}{dx} = y^2$$

$$\therefore (1 - xy)\frac{dy}{dx} = y^2$$

$$\therefore y\prime(1 - xy) = y^2$$

$\therefore$ Given function is a solution of the given differential equation.

Exercise 8.1 | Q 2.2 | Page 162

In the following example, verify that the given function is a solution of the corresponding differential equation.

Solution	D.E.
y = xn	$x^2\frac{d^2y}{dx^2} - n \times \frac{xdy}{dx} + ny = 0$

Solution:

y = x ^{n}

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = nx^{n-1}$$

Again, differentiating w.r.t. x, we get

$$\frac{d^2y}{dx^2} = n(n-1)x^{n-2}$$

$$\therefore \quad x^2\frac{d^2y}{dx^2} - nx\frac{dy}{dx} + ny$$

= $n(n-1)x^2x^{n-2}$ - $nx.nx^{n-1}$+ nx^n

= $n(n-1)x^n$ - $n^2\, x^n$ + nx^n

$=[n(n-1)-n^2+n]x^n$

$= 0$

$$\therefore x^2\frac{d^2y}{dx^2}-nx\frac{dy}{dx}+ny=0$$

∴ Given function is a solution of the given differential equation.

Exercise 8.1 | Q 2.3 | Page 162

In each of the following examples, verify that the given function is a solution of the corresponding differential equation.

Solution	D.E.
$y = e^x$	$\frac{dy}{dx}=y$

Solution:

$y = e^x$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx}=e^x$$

$$\therefore \frac{dy}{dx}=y$$

∴ Given function is a solution of the given differential equation.

Exercise 8.1 | Q 2.4 | Page 162

Determine the order and degree of the following differential equations.

Solution	D.E.
$y = 1 - \log x$	$x^2\frac{d^2y}{dx^2}=1$

Solution:

y = 1 – log x

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = -\frac{1}{x}$$

Again, differentiating w.r.t. x, we get

$$\frac{d^2y}{dx^2} = \frac{1}{x^2}$$

$$\therefore x^2 \frac{d^2y}{dx^2} = 1$$

∴ Given function is a solution of the given differential equation.

Exercise 8.1 | Q 2.5 | Page 162

Determine the order and degree of the following differential equations.

Solution	D.E
$y = ae^x + be^{-x}$	$\frac{d^2y}{dx^2} = 1$

Solution: $y = ae^x + be^{-x}$ ......(1)

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = ae^x - be^{-x}$$

$$\frac{dy}{dx} = ae^{x} - be^{-x}$$

Again, differentiating w.r.t. x, we get

$$\frac{d^2y}{dx^2} = ae^{x} - be^{-x}$$

$\therefore \frac{d^2y}{dx^2} = y$ .....[From (i)]

∴ Given function is a solution of the given differential equation.

Exercise 8.1 | Q 2.6 | Page 162

Determine the order and degree of the following differential equations.

Solution	D.E.
ax2 + by2 = 5	$xy\frac{d^2y}{dx^2} + x\left(\frac{dy}{dx}\right)^2 = y\frac{dy}{dx}$

Solution:

$ax^2 + by^2 = 5$

Differentiating w.r.t. x, we get

$2ax + 2by\frac{dy}{dx} = 0$ ....(i)

Again, differentiating w.r.t. x, we get

$2a + 2b\left(\frac{dy}{dx}\right)^2 + 2by\left(\frac{d^2y}{dx^2}\right) = 0$ ....(ii)

From (i), we get

$$a = -\frac{by}{x}\left(\frac{dy}{dx}\right)$$

Substituting the value of a in (ii), we get

$$-2\frac{by}{x}\left(\frac{dy}{dx}\right) + 2b\left(\frac{dy}{dx}\right)^2 + 2by\left(\frac{d^2y}{dx^2}\right) = 0$$

$$\therefore -\frac{y}{x}\left(\frac{dy}{dx}\right) + \left(\frac{dy}{dx}\right)^2 + y\left(\frac{d^2y}{dx^2}\right) = 0$$

$$\therefore -y\left(\frac{dy}{dx}\right) + x\left(\frac{dy}{dx}\right)^2 + xy\left(\frac{d^2y}{dx^2}\right) = 0$$

$$\therefore xy\left(\frac{d^2y}{dx^2}\right) + x\left(\frac{dy}{dx}\right)^2 = y\left(\frac{dy}{dx}\right)$$

∴ Given function is a solution of the given differential equation.

EXERCISE 8.2 [PAGE 163]

Exercise 8.2 | Q 1.1 | Page 163

Obtain the differential equation by eliminating arbitrary constants from the following equations.

$y = Ae^{3x} + Be^{-3x}$

Solution:

$y = Ae^{3x} + B.e^{-3x}$ ......(i)

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = 3Ae^{3x} - 3Be^{-3x}$$

Again, differentiating w.r. t. x, we get

$$\frac{d^2y}{dx^2} = 3A\frac{d}{dx}e^{3x} - 3B\frac{d}{dx}\left(e^{-3x}\right)$$

= $3A\left(3e^{3x}\right) - 3b\left(-3e^{-3x}\right)$

=$9Ae^{3x} + 9Be^{-3x}$

=$9\left(Ae^{3x} + Be^{-3x}\right) = 9y$ ......[From(i)]

$$\therefore \frac{d^2y}{dx^2} = 9y$$

Exercise 8.2 | Q 1.2 | Page 163

Obtain the differential equations by eliminating arbitrary constants from the following equation.

$$y = c_2 + \frac{c_1}{x}$$

Solution:

$$y = c_2 + \frac{c_1}{x}$$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{-c_1}{x^2}$$

$$\therefore x^2\frac{dy}{dx} = -c_1$$

Again, differentiating w.r.t. x, we get

$$x^2\frac{d^2y}{dx^2} + \frac{dy}{dx}.2x = 0$$

$$\therefore x^2\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} = 0$$

Exercise 8.2 | Q 1.3 | Page 163

Obtain the differential equation by eliminating arbitrary constants from the following equations.

$y = (c_1 + c_2 x) e^x$

Solution: $y = (c_1 + c_2 x) e^x$

$\therefore ye^{-x} = c_1 + c_2x$

Differentiating w.r.t. x, we get

$$y(-e^{-x}) + e^{-x}\frac{dy}{dx} = 0 + c_2$$

$$\therefore e^{-x}\left(\frac{dy}{dx} - y\right) = c_2$$

Again, differentiating w.r.t. x, we get

$$e^{-x}\left(\frac{d^2y}{dx^2} - \frac{dy}{dx}\right) - e^{-x}\left(\frac{dy}{dx} - y\right) = 0$$

$$\therefore e^{-x}\left(\frac{d^2y}{dx^2} - \frac{dy}{dx} - \frac{dy}{dx} + y\right) = 0$$

$$\therefore \frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = 0$$

Exercise 8.2 | Q 1.4 | Page 163

Obtain the differential equations by eliminating arbitrary constants from the following equations.

$y = c_1e^{3x} + c_2e^{2x}$

Solution: $y = c_1e^{3x} + c_2e^{2x}$

Dividing throughout by e^{2x}, we get

$ye^{-2x} = c_1e^{x} + c_2$

Differentiating w.r.t. x, we get

$$-2ye^{-2}x + e^{-2}x\frac{dy}{dx} = c_1e^x$$

$$\therefore e^{-2}x\left(\frac{dy}{dx} - 2y\right) = c_1e^x$$

Dividing throughout by e^x , we get

$$e^{-3}x\left(\frac{dy}{dx} - 2y\right) = c_1$$

Again, differentiating w.r.t. x, we get

$$e^{-3}x\left(\frac{d^2y}{dx^2} - 2\frac{dy}{dx}\right) - 3e^{-3x}\left(\frac{dy}{dx} - 2y\right) = 0$$

$$\therefore e^{-(3x)}\left(\frac{d^2y}{dx^2} - 2\frac{dy}{dx} - 3\frac{dy}{dx} + 6y\right) = 0$$

$$\therefore \frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 0$$

Exercise 8.2 | Q 1.5 | Page 163

Obtain the differential equation by eliminating arbitrary constants from the following equations.

$y^2 = (x + c)^3$

Solution: $y^2 = (x + c)^3$ ....(i)

Differentiating w.r.t. x, we get

$$2y\frac{dy}{dx} = 3(x + c)^2 \text{ ....(ii)}$$

Dividing (i) by (ii), we get

$$2y\frac{dy}{dx} = 3(x+c)^2$$ ....(ii)

Dividing (i) by (ii), we get

$$\frac{y^2}{2y\left(\frac{dy}{dx}\right)} = \frac{(x+c)^3}{3(x+c)^2}$$

$$\therefore \frac{y}{2\left(\frac{dy}{dx}\right)} = \frac{x+c}{3}$$

$$\therefore x + c = \frac{3y}{2\left(\frac{dy}{dx}\right)}$$

$$\therefore c = -x + \frac{3y}{2\left(\frac{dy}{dx}\right)}$$

Substituting the value of c in (i), we get

$$y^2 = \left[X + \left(-x + \frac{3y}{2\left(\frac{dy}{dx}\right)}\right)\right]^3$$

$$= \left(\frac{3y}{2\left(\frac{dy}{dx}\right)}\right)^3$$

$$\therefore y^2 = \frac{27y^3}{8\left(\frac{dy}{dx}\right)^3}$$

$$\therefore \left(\frac{dy}{dx}\right)^3 = \frac{27y}{8}$$

$$\therefore \frac{dy}{dx} = \frac{3}{2}3\sqrt{y}$$

Exercise 8.2 | Q 2 | Page 163

Find the differential equation by eliminating arbitrary constants from the relation

$x^2 + y^2 = 2ax$

Solution: Given relation is

$x^2 + y^2 = 2ax$ ...(i)

Differentiating w.r.t. x, we get

$$2x + 2y\frac{dy}{dx} = 2a \text{ ....(ii)}$$

Substituting (ii) in (i), we get

$$x^2 + y^2 = \left(2x + 2y\frac{dy}{dx}\right)x$$

$$\therefore x^2 + y^2 = 2x^2 + 2xy\frac{dy}{dx}$$

$$\therefore 2xy\frac{dy}{dx} = y^2 - x^2$$, which is the required differential equation.

Exercise 8.2 | Q 3 | Page 163

Form the differential equation by eliminating arbitrary constants from the relation

bx + ay = ab.

Solution: Given relation is

bx + ay = ab

Differentiating w.r.t. x, we get

$$b + a\frac{dy}{dx} = 0$$

Again, differentiating w.r.t. x, we get

$$a\frac{d^2y}{dx^2}=0$$

$\therefore \frac{d^2y}{dx^2}=0$, which is the required differential equation.

Exercise 8.2 | Q 4 | Page 163

Find the differential equation whose general solution is

$x^3 + y^3 = 35ax$.

Solution:

$x^3 + y^3 = 35ax$ ...(i)

Differentiating w.r.t. x, we get

$3x^2+3y^2\frac{dy}{dx}=35a$ ...(ii)

Substituting (ii) in (i), we get

$$x^3+y^3=\left(3x^2+3y^2\frac{dy}{dx}\right)x$$

$$\therefore x^3+y^3=3x^3+3x\cdot y^2\frac{dy}{dx}$$

$\therefore 2x^3-y^3+3xy^2\frac{dy}{dx}=0$, which is the required differential equation.

Exercise 8.2 | Q 5 | Page 163

Form the differential equation from the relation

$x^2 + 4y^2 = 4b^2$

Solution:

Given relation is

$x^2 + 4y^2 = 4b^2$

Differentiating w.r.t. x, we get

$$2x + 4.2y\frac{dy}{dx} = 0$$

$\therefore x + 4y\frac{dy}{dx} = 0$, which is the required differential equation.

EXERCISE 8.3 [PAGE 165]

Exercise 8.3 | Q 1.1 | Page 165

Solve the following differential equation.

$$\frac{dy}{dx} = x^2y + y$$

Solution:

$$\frac{dy}{dx} = x^2y + y = (x^2 + 1)y$$

$$\therefore \frac{1}{y}dy = (x^2 + 1)dx$$

Integrating on both sides, we get

$$\int \frac{1}{y}dy = \int (x^2 + 1)dx$$

$$\therefore \log |y| = \frac{x^3}{3} + x + c$$

Exercise 8.3 | Q 1.2 | Page 165

Solve the following differential equation.

$$\frac{d\theta}{dt} = -k(\theta - \theta_0)$$

Solution:

$$\frac{d\theta}{dt} = -k(\theta - \theta_0), \text{ k is constant.}$$

$$\therefore \frac{d\theta}{\theta - \theta_0} = -kdt$$

Integrating on both sides, we get

$$\int \frac{d\theta}{\theta - \theta_0} = -k\int dt$$

$\therefore \log|\theta - \theta_0| = - kt + c$

$\therefore \theta - \theta_0 = e^{kt+c}$

Exercise 8.3 | Q 1.3 | Page 165

Solve the following differential equation

$(x^2 - yx^2)\, dy + (y^2 + xy^2)\, dx = 0$

Solution:

$(x^2 - yx^2)\, dy + (y^2 + xy^2)\, dx = 0$

$\therefore x^2 (1 - y)\, dy = - y^2 (1 + x)\, dx$

$$\therefore \left(\frac{1-y}{y^2}\right)dy = -\left(\frac{1+x}{x^2}\right)dx$$

Integrating on both sides, we get

$$\int\left(\frac{1}{y^2}-\frac{1}{y}\right)dy=-\int\left(\frac{1}{x^2}+\frac{1}{x}\right)dx$$

$$\therefore -\frac{1}{y}-\log|y|=-\left(-\frac{1}{x}+\log\ |x|\right)+c$$

$$\therefore \frac{-1}{y}-\log|y|=\frac{1}{x}-\log\ |x|+c$$

$$\therefore \log\ |x|-\log|y|=\frac{1}{x}+\frac{1}{y}+c$$

Exercise 8.3 | Q 1.4 | Page 165

Solve the following differential equation.

$$y^3-\frac{dy}{dx}=x\frac{dy}{dx}$$

Solution:

$$y^3-\frac{dy}{dx}=x\frac{dy}{dx}$$

$$\therefore y^3=(1+x)\frac{dy}{dx}$$

$$\therefore \frac{dx}{(1+x)}=\frac{dy}{y^3}$$

Integrating on both sides, we get

$$\int\frac{dx}{1+x}=\int\frac{dy}{y^3}$$

$$\therefore \log|1+x|=-\frac{1}{2y^2}+c$$

$\therefore 2y^2 \log | 1 + x | = - 1 + 2y^2c$

Exercise 8.3 | Q 2.1 | Page 165

For each of the following differential equations find the particular solution.

$(x - y^2 x)\,dx - (y + x^2 y)\,dy = 0$, when $x = 2$, $y = 0$

Solution:

$(x - y^2 x)dx - (y + x^2 y)\,dy = 0$, when $x = 2$, $y = 0$

$\therefore x(1 - y^2)\,dx = y(1 + x^2)\,dy$

$$\therefore \frac{xdx}{1+x^2} = \frac{ydy}{1-y^2}$$

Integrating on both sides, we get

$$\int \frac{2x}{1+x^2}dx = \int \frac{2y}{1-y^2}dy$$

$$\therefore \int \frac{2x}{1+x^2}dx = -\int \frac{-2y}{1-y^2}dy$$

$$\therefore \log\left|1+x^2\right| = -\log\left|1-y^2\right| + \log|c|$$

$$\therefore \log\left|1+x^2\right| = \log\left|\frac{c}{1-y^2}\right|$$

$\therefore (1 + x^2)(1 - y^2) = c$ ...(i)

When $x = 2$, $y = 0$, we have

$(1 + 4)(1 - 0) = c$

$\therefore c = 5$

Substituting $c = 5$ in (i),we get

$(1 + x^2)(1 - y^2) = 5$,

which is the required particular solution.

Exercise 8.3 | Q 2.2 | Page 165

For each of the following differential equations find the particular solution.

$$(x+1)\frac{dy}{dx}-1=2e^{-y},$$

when y = 0, x = 1

Solution:

$$(x+1)\frac{dy}{dx}-1=2e^{-y}$$

$$\therefore\ (x+1)\frac{dy}{dx}=\frac{2}{e^{y}}+1$$

$$\therefore\ (x+1)\frac{dy}{dx}=\frac{(2e^{y})}{e^{y}}$$

$$\therefore\ \frac{e^{y}}{2+e^{y}}dy=\frac{dx}{1+x}$$

Integrating on both sides, we get

$$\int\frac{e^{y}}{2+e^{y}}dy=\int\frac{dx}{1+x}$$

$\therefore$ log| $2+e^{y}$| = log |1+x| + log |c|

$\therefore$ log |$2+e^{y}$| = log |c(1+x)|

$\therefore$ $2+e^{y}$ = c (1+x) ......(i)

When y = 0, x = 1, we have

$2 + e^{0} = c\ (1 + 1)$

$\therefore$ 2 + 1 = 2c

$\therefore \quad c = \frac{3}{2}$

Substituting $c = \frac{3}{2}$ in (i), we get

$$2 + e^y = \frac{3}{2}(1 + x)$$

$\therefore\ 4 + 2e^y = 3 + 3x$

$\therefore\ 3x - 2e^y - 1 = 0$, which is the required particular solution

Exercise 8.3 | Q 2.3 | Page 165

For each of the following differential equations find the particular solution.

$$y(1 + \log x)\frac{dx}{dy} - x \log x = 0,$$

when $x=e$, $y = e^2$.

Solution:

$$y(1 + \log x)\frac{dx}{dy} - x \log x = 0,$$

when $x=e$, $y = e^2$.

$$\therefore\ y(1 + \log x)\frac{dx}{dy} = x \log x$$

$\therefore\ y(1 + \log x)dx = x \log x\ dy$

$$\therefore\ \frac{1}{y}dy = \frac{1 + \log x}{x \log x}\ dx$$

Integrating on both sides, we get

$$\int \frac{1}{y} dy = \int \frac{1 + \log x}{x \log x} dx$$

$\therefore$ log |y| = log |x log x| + log |c|

$\therefore$ log |y| = log |cx log x|

When x = e, y = e^2 , we have

$\therefore$ y = cx log x ...(i)

e^2 = ce log e

$\therefore$ e^2 = ce

$\therefore$ c = e

Substituting c = e in (i), we get

y = ex log x, which is the required particular solution.

Exercise 8.3 | Q 2.4 | Page 165

For the following differential equation find the particular solution.

dy/ dx = (4x + y + 1),

when y = 1, x = 0

Solution:

$$\frac{dy}{dx} = (4x + y + 1) \text{ ..(i)}$$

Put 4x + y + 1 = t ...(ii)

Differentiating w.r.t. x, we get

$$4+\frac{dy}{dx}=\frac{dt}{dx}$$

$$\therefore \frac{dy}{dx}=\frac{dt}{dx}-4 \text{ .... (iii)}$$

Substituting (ii) and (iii) in (i), we get

$$\frac{dt}{dx}-4=t$$

$$\therefore \frac{dt}{dx}=t+4$$

$$\therefore \frac{dt}{t+4}=dx$$

Integrating on both sides, we get

$$\int \frac{dt}{t+4}=\int dx$$

∴ log | t + 4 | = x + c

∴ log |(4x + y + 1) + 4 | = x + c

∴ log | 4x + y + 5 | = x + c …(iv)

When y = 1, x = 0, we have

log |4(0) + 1 + 5| = 0 + c

∴ c = log |6|

Substituting c = log |6| in (iv), we get

log |4x + y + 5| = x + log |6|

∴ log |4x + y + 5 | - log |6| = x

$$\therefore \log\left|\frac{4x+y+5}{6}\right|=x,$$

which is the required particular solution.

EXERCISE 8.4 [PAGE 167]

Exercise 8.4 | Q 1.1 | Page 167

Solve the following differential equation.

xdx + 2y dy = 0

Solution:

xdx + 2y dy = 0

Integrating on both sides, we get

$$\int x dx + 2\int y\, dy = 0$$

$$\therefore \frac{x^2}{2} + \frac{2y^2}{2} = c_1$$

$\therefore x^2 + 2y^2 = c$ ....$[2c_1 = c]$

Exercise 8.4 | Q 1.2 | Page 167

Solve the following differential equation.

y^2 dx + (xy + x^2) dy = 0

Solution:

y^2 dx + (xy + x^2) dy = 0

$\therefore$ (xy + x^2) dy = - y^2 dx

$$\therefore \frac{dy}{dx} = \frac{y^2}{xy + x^2} \quad ...(i)$$

Put y = tx ...(ii)

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = t + x\frac{dt}{dx} \quad \text{...(iii)}$$

Substituting (ii) and (iii) in (i), we get

$$t + x\frac{dt}{dx} = \frac{-t^2x^2}{x.tx + x^2}$$

$$\therefore t + x\frac{dt}{dx} = \frac{-t^2x^2}{tx^2 + x^2}$$

$$\therefore \; t + x\frac{dt}{dx} = \frac{-t^2x^2}{x^2(t+1)}$$

$$\therefore \; x\frac{dt}{dx} = \frac{-t^2}{t+1} - t$$

$$\therefore x\frac{dt}{dx} = \frac{-t^2 - t^2 - t}{t+1}$$

$$\therefore x\frac{dt}{dx} = \frac{\left(-2t^2 + t\right)}{t+1}$$

$$\therefore \frac{t+1}{2t^2+t}dt = -\frac{1}{x}dx$$

Integrating on both sides, we get

$$\int \frac{t+1}{2t^2+t}dt = -\int \frac{1}{x}dx$$

$$\therefore \int \frac{2t+1-t}{t(2t+1)}dt = -\int \frac{1}{x}dx$$

$$\therefore \int \frac{1}{t} dt - \int \frac{1}{2t+1} dt = -\int \frac{1}{x} dx$$

$\therefore$ log | t | - $\frac{1}{2}$ log |2t + 1| = - log |x| + log |c|

$\therefore$ 2log| t | - log |2t + 1| = - 2log |x| + 2 log |c|

$$\therefore 2\log\left|\frac{y}{x}\right| - \log\left|\frac{2y}{x} + 1\right| = -2\log|x| + 2\log|c|$$

$\therefore$ 2log |y| - 2log |x| - log |2y + x| + log |x|

= -2log |x| + 2log |c|

$\therefore$ log $|y^2|$ + log |x| = log $|c^2|$ + log |2y + x|

$\therefore$ log $|y^2 x|$ = log $|c^2 (x + 2y)|$

$\therefore$ xy 2 = c^2 (x + 2y)

Exercise 8.4 | Q 1.3 | Page 167

Solve the following differential equation.

$x^2y\, dx - (x^3 + y^3)\, dy = 0$

Solution:

$x^2y\, dx - (x^3 + y^3)\, dy = 0$

$\therefore x^2y\, dx = (x^3 + y^3)\, dy$

$$\therefore \frac{dy}{dx} = \frac{x^2 y}{x^3 + y^3} \quad \text{...(i)}$$

Put y = tx ...(ii)

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = t + x\frac{dt}{dx} \quad ...(iii)$$

Substituting (ii) and (iii) in (i), we get

$$t + x\frac{dt}{dx} = \frac{x^2.tx}{x^3 + t^3x^3}$$

$$\therefore t + x\frac{dt}{dx} = \frac{x^3.t}{x^3(1+t^3)}$$

$$\therefore x\frac{dt}{dx} = \frac{t}{1+t^3} - t$$

$$\therefore x\frac{dt}{dx} = \frac{t - t - t^4}{1+t^3}$$

$$\therefore x\frac{dt}{dx} = -\frac{t^4}{1+t^3}$$

$$\therefore \frac{1+t^3}{t^4}\,dt = -\frac{dx}{x}$$

Integrating on both sides, we get

$$\int \frac{1+t^3}{t^4} = -\int \frac{1}{x}dx$$

$$\therefore \int \left(\frac{1}{t^4} + \frac{1}{t}\right)dt = -\int \frac{1}{x}dx$$

$$\therefore \int t^{-4}dt + \int \frac{1}{t}dt = -\int \frac{1}{x}dx$$

$$\therefore \frac{t^{-3}}{-3} + \log|t| = -\log|x| + \log|c_1|$$

$$\therefore -\frac{1}{-3}t^3 + \log|t| = -\log|x| + \log|c_1|$$

$$\therefore -\frac{1}{3} \cdot \frac{1}{\left(\frac{y}{x}\right)^3} + \log\left|\frac{y}{x}\right| = -\log|x| + \log|c_1|$$

$$\therefore -\frac{x^3}{3y^3} + \log|y| - \log|x| = -\log|x| + \log|c_1|$$

$$\therefore \log|y| + \log|c| = \frac{x^3}{3y^3} \ldots\ldots\ldots [-\log|c_1| = \log|c|]$$

$$\therefore \log|yc| = \frac{x^3}{3y^3}$$

Exercise 8.4 | Q 1.4 | Page 167

Solve the following differential equation.

$$\frac{dy}{dx} + \frac{x - 2y}{2x - y} = 0$$

Solution:

$$\frac{dy}{dx} + \frac{x - 2y}{2x - y} = 0 \text{ ....(i)}$$

Put y = tx ...(ii)

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = t + x\frac{dt}{dx} \text{...(iii)}$$

Substituting (ii) and (iii) in (i), we get

$$t + x\frac{dt}{dx} + \frac{x - 2tx}{2x - tx} = 0$$

$$\therefore x\frac{dt}{dx} + t + \frac{1 - 2t}{2} - t = 0$$

$$\therefore x\frac{dt}{dx} + \frac{2t - t^2 + 1 - 2t}{2} - t = 0$$

$$\therefore x\frac{dt}{dx} + \frac{1 - t^2}{2 - t} = 0$$

$$\therefore x\frac{dt}{dx} = -\frac{1 - t^2}{2 - t}$$

$$\therefore = \frac{2 - t}{1 - t^2}dt = \frac{dx}{x}$$

$$\therefore \frac{2 - t}{t^2 - 1}dt = \frac{dx}{x}$$

Integrating on both sides, we get

$$\int \frac{2 - t}{t^2 - 1}dt = \int \frac{dx}{x}$$

$$\therefore \int \frac{2 - t}{(t + 1)(t - 1)}dt = \int \frac{dx}{x}$$

Let $2 - \frac{t}{(t + 1)(t - 1)} = \frac{A}{t + 1} + \frac{B}{t - 1}$

∴ 2 - t = A(t -1) + B(t + 1)

Putting t = 1, we get

∴ 2 -1 = A(1 -1) + B(1 + 1)

∴ B = $\frac{1}{2}$

Putting t = -1, we get

2 -(-1) = A(-1 -1) + B(-1 + 1)

∴ A = $\frac{-3}{2}$

$$\therefore \int \frac{-\frac{3}{2}}{t+1} dt + \int \frac{\frac{1}{2}}{t-1} dt = \int \frac{dx}{x}$$

$$\therefore \frac{-3}{2} \int \frac{1}{t+1} dt + \frac{1}{2} \int \frac{1}{t-1} dt = \int \frac{dx}{x}$$

$$\therefore \frac{-3}{2} \log|t+1| + \frac{1}{2} \log|t-1| = \log|x| + \log|c_1|$$

$$\therefore -3 \log\left|\frac{y+x}{x}\right| + \log\left|\frac{y-x}{x}\right| = 2\log|x| + 2\log|c_1|$$

$\therefore$ -3 log |y+x| + 3 log |x| + log | y -x| - log |x|

= 2 log |x| + 2 log $|c_1|$

$\therefore$ log |y - x| = 3 log |y+x|+ 2 log $|c_1|$

$\therefore$ log |y- x |= log $|(y+ x)^3|$+ log $| c_1^2|$

$\therefore$ log | y - x| = log $| c_1^2 (x+y)^3|$

$\therefore$ (y - x) = c(x + y) ^{3} ... $|c_1^2 c|$

NOTES

Answer given in the textbook is $\log\left|\frac{x+y}{x-y}\right| - \frac{1}{2}\log|x^2 - y^2| + 2\log x = \log c.$

However, as per our calculation it is '(y -x) = c$(x+y)^3$.

Exercise 8.4 | Q 1.5 | Page 167

Solve the following differential equation.

$(x^2 - y^2)$ dx + 2xy dy = 0

Solution:

$(x^2 - y^2)\,dx + 2xy\,dy = 0$

$\therefore 2xy\,dy = (y^2 - x^2)\,dx$

$$\therefore \frac{dy}{dx} = \frac{y^2 - x^2}{2xy} \ldots\ldots(i)$$

Put $y = tx$ ...(ii)

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = t + x\frac{dt}{dx} \ \ldots(iii)$$

Substituting (ii) and (iii) in (i), we get

$$t + x\frac{dt}{dx} = \frac{t^2x^2 - x^2}{2tx^2}$$

$$\therefore x\frac{dt}{dx} = \frac{t^2 - 1}{2t} - t = \frac{-(1 + t^2)}{2t}$$

$$\therefore 2\frac{t}{1 + t^2}dt = -\frac{dx}{x}$$

Integrating on both sides, we get

$$\int 2\frac{t}{1 + t^2}dt = -\int \frac{dx}{x}$$

$\therefore \log|1 + t^2| = -\log|x| + \log|c|$

$$\therefore \log\left|1 + \frac{y^2}{x^2}\right| = \log\left|\frac{c}{x}\right|$$

$$\therefore \frac{x^2 + y^2}{x^2} = \frac{c}{x}$$

$\therefore\ x^2 + y^2 = cx$

Exercise 8.4 | Q 1.6 | Page 167

Solve the following differential equation.

$$xy\frac{dy}{dx} = x^2 + 2y^2$$

Solution:

$$xy\frac{dy}{dx} = x^2 + 2y^2$$

$$\therefore \frac{dy}{dx} = \frac{x^2 + 2y^2}{xy} \ldots(i)$$

Put y = tx ...(ii)

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = t + x\frac{dt}{dx} \quad \text{...(iii)}$$

Substituting (ii) and (iii) in (i), we get

$$t + x\frac{dt}{dx} = \frac{x^2 + 2t^2x^2}{x(tx)}$$

$$\therefore t + x\frac{dt}{dx} = \frac{x^2\left(1 + 2t^2\right)}{x^2 t}$$

$$\therefore x\frac{dt}{dx} = \frac{1 + 2t^2}{t} - t = \frac{1 + t^2}{t}$$

$$\therefore \frac{t}{1 + t^2}dt = \frac{1}{x}dx$$

Integrating on both sides, we get

$$\frac{1}{2}\int \frac{2t}{1 + t^2}dt = \int \frac{dx}{x}$$

$\therefore \log|1 + t^2| = 2\log|x| + 2\log|c_1|$

$= \log|x^2| + \log|c|$ ...$[\log c_1^2 = \log c]$

$\therefore \log|1 + t^2| = \log|cx^2|$

$\therefore 1 + t^2 = cx^2$

$$\therefore 1 + \frac{y^2}{x^2} = cx^2$$

$\therefore x^2 + y^2 = cx^4$

Exercise 8.4 | Q 1.7 | Page 167

Solve the following differential equation.

$$x^2\frac{dy}{dx} = x^2 + xy - y^2$$

Solution:

$$x^2\frac{dy}{dx} = x^2 + xy - y^2$$

$$\therefore \frac{dy}{dx} = \frac{x^2 + xy - y^2}{x^2} \quad \text{...(i)}$$

Put y = tx ...(ii)

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = t + x\frac{dt}{dx} \quad \text{...(iii)}$$

Substituting (ii) and (iii) in (i), we get

$$t + x\frac{dt}{dx} = \frac{x^2 + x(tx) - (tx)^2}{x^2}$$

$$\therefore t + x\frac{dt}{dx} = \frac{x^2 + tx^2 - t^2x^2}{x^2}$$

$$\therefore t + x\frac{dt}{dx} = 1 + t - t^2$$

$$\therefore \ x\frac{dt}{dx} = 1 + t - t^2$$

$$\therefore \ x\frac{dt}{dx} = 1 - t^2$$

$$\therefore \ \frac{dt}{1^2 - t^2} = \frac{dx}{x}$$

Integrating on both sides, we get

$$\int \frac{dt}{(1)^2 - (t)^2} = \int \frac{dx}{x}$$

$$\therefore \frac{1}{2(1)} \ \log \left|\frac{1+t}{1-t}\right| = \log\ |x| + \log|c_1| \ ...[Q \int \frac{dx}{a^2 - x^2} = \frac{1}{2a} \ \log\left|\frac{a+x}{a-x}\right| + c]$$

$$\therefore \log\left|\frac{1+t}{1-t}\right| = \log|x| + \log|c_1|$$

$$\therefore \log\left|\frac{1+t}{1-t}\right| = \log|c_1^2 x^2|$$

$$\therefore \frac{1 + \left(\frac{y}{x}\right)}{1 - \left(\frac{y}{x}\right)} = c_1^2 x^2$$

$$\therefore \frac{x+y}{x-y} = cx^2 \ldots [c_1^2 = c]$$

EXERCISE 8.5 [PAGE 168]

Exercise 8.5 | Q 1.1 | Page 168

Solve the following differential equation.

$$\frac{dy}{dx} + y = e^{-x}$$

Solution:

$$\frac{dy}{dx} + y = e^{-x}$$

The given equation is of the form

$$\frac{dy}{dx} + py = Q$$

where, P = 1 and Q = e^{-X}

$\therefore$ I.F. = $e\int^{pdx} = e\int^{1.dx} = e^x$

$\therefore$ Solution of the given equation is

$$y(I.F.) = \int Q(I.F.)dx + c$$

$$\therefore\ ye^x = \int e^{-x}e^x dx + c$$

$$\therefore\ ye^x = \int 1dx + c$$

$\therefore$ y e^X = x+c

Exercise 8.5 | Q 1.2 | Page 168

Solve the following differential equation.

$$\frac{dy}{dx} + y = 3$$

Solution:

$$\frac{dy}{dx} + y = 3$$

The given equation is of the form

$$\frac{dy}{dx} + py = Q$$

where, P = 1 and Q = 3

∴ I.F. = $e\int^{pdx} = e\int^{1dx} = e^x$

∴ Solution of the given equation is

$$y(I.F.) = \int Q(I.F.)dx + c$$

$$\therefore ye^x = \int 3e^x dx + c$$

∴ $ye^x = 3e^x + c$

Exercise 8.5 | Q 1.3 | Page 168

Solve the following differential equation.

$$x\frac{dy}{dx} + 2y = x^2 \log x$$

Solution:

$$x\frac{dy}{dx} + 2y = x^2 \log x$$

Dividing throughout by x, we get

$$\frac{dy}{dx} + \frac{2}{x}y = x \log x$$

The given equation is of the form

$$\frac{dy}{dx} + py = Q$$

where, P = $\frac{2}{x}$ and $Q = x \log x$

∴ I.F. =$e^{\int pdx} = e^{2\int \frac{1}{x}dx} = e^{2\log|x|} = e^{\log|x^2|} = x^2$

∴ Solution of the given equation is

$$y(I.F.) = \int Q(I.F.)dx + c$$

$$\therefore yx^2 = \int (x \log x)x^2 dx + c$$

$$= \int x^3 \log x dx + c$$

$$= \log x \int x^3 dx - \int \left(\frac{d}{dx} \log x \int x^3 dx \right) dx + c$$

$$= \frac{x^4}{4} \log x - \int \frac{1}{x} \left(\frac{x^4}{4} \right) dx + c$$

$$= \frac{x^4}{4} \log x - \frac{1}{4} \int x^3 dx + c$$

$$\therefore\ yx^2 = \frac{x^4}{4} \log x - \frac{X^4}{16} + c$$

Exercise 8.5 | Q 1.4 | Page 168

Solve the following differential equation.

$$(x+y)\frac{dy}{dx} = 1$$

Solution:

$$(x+y)\frac{dy}{dx}=1$$

$$\therefore \frac{dy}{dx}=\frac{1}{x+y}$$

$$\therefore \frac{dy}{dx}=(x+y)$$

$$\therefore \frac{dx}{dy}-x=y$$

The given equation is of the form $\frac{dx}{dy}+Px=Q$

where, P = -1 and Q = y

$$\therefore I.F.=e\int^{pdy}=e\int^{-1dy}=e^{-y}$$

$\therefore$ Solution of the given equation is

$$x(I.F.)=\int Q(I.F.)dy+c$$

$$\therefore xe^{-y}=\int ye^{-y}dy+c$$

$$\therefore xe^{-y}=y\int e^{-y}dy-\int\left[\frac{d}{dy}(y)\int e^{-y}dy\right]dy+c$$

$$\therefore xe^{-y}=-y(e^{-y})-\int 1\times(-e^{-y})dy+c$$

$$\therefore xe^{-y}=-ye^{-y}-e^{-y}+c$$

$\therefore$ x = -y -1 + c e^y

$\therefore\ x + y + 1 = c\,e^y$

Exercise 8.5 | Q 1.5 | Page 168

Solve the following differential equation.

$y\,dx + (x - y^2)\,dy = 0$

Solution:

$y\,dx + (x - y^2)\,dy = 0$

$\therefore y\,dx = (y^2 - x)\,dy$

$$\therefore \frac{dx}{dy} = \frac{y^2 - x}{y} = y - \frac{x}{y}$$

$$\therefore \frac{dx}{dy} + \frac{x}{y} = y$$

The given equation is of the form

$$\frac{dx}{dy} + Px = Q$$

where, P = $\frac{1}{y}$ and Q = y

$$\therefore \text{I.F.} = e\int^{pdy} = e\int^{\frac{1}{y}dy} = e^{\log|y|} = y$$

$\therefore$ Solution of the given equation is

$$x(I.F.) = \int Q(I.F.)dy + c_1$$

$$\therefore\ xy = \int y(y)dy = \frac{y^3}{3} + c_1$$

$\therefore\ 3xy = y^3 + c \quad ...[3c_1 = c]$

Exercise 8.5 | Q 1.6 | Page 168

Solve the following differential equation.

$$\frac{dy}{dx} + 2xy = x$$

Solution:

$$\frac{dy}{dx} + 2xy = x$$

The given equation is of the form

$$\frac{dy}{dx} + py = Q$$

where, P = 2x and Q = x

$$\therefore I.F. = e^{\int P dx} = e^{\int 2x\, dx = e^{x^2}}$$

∴ Solution of the given equation is

y(I.F.) = $\int Q(I.F.)dx + c$

$$\therefore ye^{x^2} \int xe^{x^2} dx + c$$

In R. H. S., put x^2 = t

Differentiating w.r.t. x, we get

2x dx = dt

$$\therefore ye^{x^2} = \int e^t \frac{dt}{2} + c$$

= $\frac{1}{2} \int e^t dt + c$

$= \frac{e^t}{2} + c$

$\therefore ye^{x^2} = \frac{1}{2}e^{x^2} + c$

Exercise 8.5 | Q 1.7 | Page 168

Solve the following differential equation.

$(x+a)\frac{dy}{dx} =- y + a$

Solution:

$(x+a)\frac{dy}{dx} =- y + a$

$\therefore \frac{dy}{dx} + \frac{y}{(x+a)} = \frac{a}{(x+a)}$

The given equation is of the form

$\frac{dy}{dx} + py = Q$

where, $P = \frac{1}{(x+a)}$ and $Q = \frac{a}{(x+a)}$

$\therefore$ I.F. = $e^{\int pdx = e \int \frac{1}{x+a} \hat{} dx}$

= $e^{\log^{|x+a|}} = (x+a)$

$\therefore$ Solution of the given equation is

$y(I.F.) = \int Q(I.F.)dx + c$

$\therefore y(x+a) = \int \frac{a}{(x+a)}(x+a)dx + c$

$$\therefore y(x+a) = a\int 1\,dx + c$$

∴ y (x + a) = ax + c

Exercise 8.5 | Q 1.8 | Page 168

Solve the following differential equation.

dr + (2r)dθ= 8dθ

Solution:

dr + (2r)dθ= 8dθ

$$\frac{dr}{d\theta} + 2r = 8$$

The given equation is of the form

$$\frac{dr}{d\theta} + Pr = Q$$

where, P = 2 and Q = 8

I.F. = $e^{\int P d_\theta} = e^{\int 2 d_\theta} = e^{2\theta}$

Solution of the given equation is

$$r(I.F.) = \int Q(I.F.)d\theta + c$$

$$re^{2\theta} = \int 8e^{2\theta}\,d\theta + c$$

$$re^{2\theta} = 8\int e^{2\theta}\,d\theta + c$$

$$re^{2\theta} = 8\frac{e^{2\theta}}{2} + c$$

$$re^{2\theta} = 4e^{2\theta} + c$$

EXERCISE 8.6 [PAGE 170]

Exercise 8.6 | Q 1 | Page 170

In a certain culture of bacteria, the rate of increase is proportional to the number present. If it is found that the number doubles in 4 hours, find the number of times the bacteria are increased in 12 hours.

Solution:

Let x be the number of bacteria in the culture at time t.

Then the rate of increase is $\frac{dy}{dx}$ which is proportional to x.

$\therefore \frac{dx}{dt} \propto x$

$\therefore \frac{dx}{dt}$ = kx, where k is a constant

$\therefore \frac{dx}{x}$ = k dt

On integrating, we get

$$\int \frac{dx}{x} = k\int 1dt + c$$

$\therefore \log x = kt + c$

Initially, i.e. when t = 0, let $x = x_0$

$\therefore \log x_0 = k \times 0 + c$

$\therefore c = \log x_0$

$\therefore$ log x = kt + log x_0

$\therefore$ log x - log x_0 = kt

$\therefore \log\left(\frac{x}{x_0}\right) = kt \quad ...(1)$

Since the number doubles in 4 hours, i.e. when t = 4,

x = $2x_0$

$\therefore \log\left(\frac{2x_0}{x_0}\right) = 4k$

$\therefore k = \frac{1}{4} \log 2$

$\therefore$ (1) becomes, $\log\left(\frac{x}{x_0}\right) = \frac{t}{4} \log 2$

When t = 12, we get

$\log\left(\frac{x}{x_0}\right) = \frac{12}{4} \log 2 = 3 \log 2$

$\therefore \log\left(\frac{x}{x_0}\right) = \log 8$

$\therefore \frac{x}{x_0} = 8$

$\therefore$ x = $8x_0$

$\therefore$ number of bacteria will be 8 times the original number in 12 hours.

Exercise 8.6 | Q 2 | Page 170

The population of a town increases at a rate proportional to the population at that time. If the population increases from 40 thousands to 60 thousands in 40 years, what will be the population in another 20 years?

(Given: $\sqrt{\frac{3}{2}} = 1.2247$)

Solution:

Let 'x' be the population at time 't'

$$\therefore \frac{dx}{dt} \propto x$$

$\therefore \frac{dx}{dt} = kx$, where k is the constant of proportionality.

$$\therefore \frac{dx}{dt} = k\,dt$$

Integrating on both sides, we get

$$\int \frac{dx}{x} = k\int 1dt$$

$\therefore$ log x = kt + c ...(i)

When t = 0, x = 40000

$\therefore$ log (40000) = k(0) + c

$\therefore$ c = log (40000)

$\therefore$ log x = kt + log (40000) ...(ii) [From (i)]

When, t = 40, x = 60000

$\therefore$ log (60000) = 40k + log (40000)

$\therefore$ 40k = log (60000) – log (40000)

$\therefore$ 40k = log $\left(\frac{60000}{40000}\right)$

$\therefore$ k = $\frac{1}{40}\log\left(\frac{3}{2}\right)$ ...(iii)

When t = 60, we get

log x = k(60) + log (40000) ...[From (ii)]

$\therefore$ $\log x = \left[\frac{1}{40}\log\left(\frac{3}{2}\right)\right](60) + \log(40000)$ ...[From (iii)]

$\therefore$ $\log x = \frac{3}{2}\log\left(\frac{3}{2}\right) + \log(40000)$

$= \frac{3}{2}\log\left(\frac{3}{2}\right)^{\frac{3}{2}} + \log(40000)$

$= \frac{3}{2}\log\left(\sqrt{\frac{3}{2}}\right)^{3} + \log(40000)$

$= \log\left(\frac{3}{2}\sqrt{\frac{3}{2}} \times \ \log 40000\right)$

$\therefore$ $\log x = \log\left(\frac{3 \times 1.2247}{2} \times 40000\right)$

$\therefore$ x = 73482

$\therefore$ Population in another 20 years, i.e., in 60 years will be 73482.

Exercise 8.6 | Q 3 | Page 170

The rate of growth of bacteria is proportional to the number present. If initially, there were 1000 bacteria and the number doubles in 1 hour, find the number of bacteria after 5/2 hours.

(Given: $\sqrt{2} = 1.414$)

Solution:

Let ‘x’ be the number of bacteria present at time ‘t’.

$$\therefore \frac{dx}{dt} \propto x$$

$\therefore \dfrac{dx}{dt} = kx$, where k is the constant of proportionality.

$$\therefore \frac{dx}{x} = k\,dt$$

Integrating on both sides, we get

$$\int \frac{dx}{x} = k \int dt$$

∴ log x = kt + c ...(i)

When t = 0, x = 1000

∴ log (1000) = k(0) + c

∴ c = log (1000)

∴ log x = kt + log (1000) ...(ii)[From (i)]

When t = 1, x = 2000

∴ log (2000) = k(1) + log (1000)

∴ log (2000) - log (1000) = k

$\therefore k = \log\left(\dfrac{2000}{1000}\right) = \log 2$ ...(iii)

When t = $\frac{5}{2}$, we get

log x = $\frac{5}{2}k + \log(1000)$ ...[From (ii)]

∴ log x = $\left(\frac{5}{2}\right) \log 2 + \log(1000)$ ...[From (iii)]

= log $\left(2^{\frac{5}{2}}\right) + \log(1000)$

= $\log\left(4\sqrt{2}\right) + \log(1000)$

= $\log\left(4000\sqrt{2}\right)$

= log (4000 x1.414)

∴ log x = log (5656)

∴ x = 5656

Thus, there will be 5656 bacteria after $\frac{5}{2}$ hours.

Exercise 8.6 | Q 4 | Page 170

Find the population of a city at any time t, given that the rate of increase of population is proportional to the population at that instant and that in a period of 40 years, the population increased from 30,000 to 40,000.

Solution: Let P be the population of the city at time t.

Then $\frac{dP}{dt}$, the rate of increase of population, is proportional to P.

∴ $\frac{dP}{dt} \propto P$

∴ $\frac{dP}{dt}$ = kP, where k is a constant.

$\therefore \dfrac{dP}{P} = k\,dt$

On integrating, we get

$$\int \frac{1}{P}dP = k\int dt + c$$

$\therefore \log P = kt + c$

Initially, i.e. when t = 0, P = 30000

$\therefore \log 30000 = k \times 0 + c \qquad \therefore c = \log 30000$

$\therefore \log P = kt + \log 30000$

$\therefore \log P - \log 30000 = kt$

$\therefore \log\left(\dfrac{P}{30000}\right) = kt \qquad .....(1)$

Now, when t = 40, P = 40000

$\therefore \log\left(\dfrac{40000}{30000}\right) = k \times 40$

$\therefore k = \dfrac{1}{40}\log\left(\dfrac{4}{3}\right)$

$\therefore$ (1) becomes, $\log\left(\dfrac{P}{30000}\right) = \dfrac{t}{40}\log\left(\dfrac{4}{3}\right) = \log\left(\dfrac{4}{3}\right)^{\frac{t}{40}}$

$\therefore \dfrac{P}{30000} = \left(\dfrac{4}{3}\right)^{\frac{t}{40}}$

$\therefore P = 30000\left(\dfrac{4}{3}\right)^{\frac{t}{40}}$

$\therefore$ the population of the city at time t = 30000 $\left(\frac{4}{3}\right)^{\frac{t}{40}}$

Exercise 8.6 | Q 5 | Page 170

The rate of depreciation dV/dt of a machine is inversely proportional to the square of t + 1, where V is the value of the machine t years after it was purchased. The initial value of the machine was ₹ 8,00,000 and its value decreased ₹1,00,000 in the first year. Find its value after 6 years.

Solution: According to the given condition,

$$\frac{dV}{dt} \propto \frac{1}{(t+1)^2}$$

$$\therefore \frac{dV}{dt} = \frac{-k}{(t+1)^2}$$ ...[Negative sign indicates disintegration]

$$\therefore dV = \frac{-kdt}{(t+1)^2}$$

Integrating on both sides, we get

$$\int dV = -k\int \frac{dt}{(t+1)^2}$$

$$\therefore V = \frac{k}{t+1} + c$$

when t = 0, V = 8,00,000

$$\therefore 8,00,000 = \frac{k}{(0+1)} + c$$

$\therefore$ 8,00,000 = k + c ...(i)

when t = 1, V = 7,00,000

$$\therefore 7,00,000 = \frac{k}{(1+1)} + c$$

$$\therefore 7,00,000 = \frac{k}{2} + c \text{ ...(ii)}$$

From (i) – (ii), we get

$$1,00,000 = \frac{k}{2}$$

∴ k = 2,00,000 ...(iii)

Substituting (iii) in (i), we get

c = 6,00,000 ...(iv)

when t = 6, we get

$$V = \frac{k}{(6+1)} + c$$

$$= \frac{2,00,000}{7} + 6,00,000$$

= 6,28,571.4286

≈6,28,571

∴ Value of the machine after 6 years is ₹ 6,28,571.

MISCELLANEOUS EXERCISE 8 [PAGES 171 - 173]

Miscellaneous Exercise 8 | Q 1.01 | Page 171

Choose the correct alternative.

The order and degree of $\left(\frac{dy}{dx}\right)^3 - \frac{d^3y}{dx^3} + ye^x = 0$ are respectively.

1. **3, 1**
2. 1, 3

3. 3, 3
4. 1, 1

Solution:

The order and degree of $\left(\frac{dy}{dx}\right)^3 - \frac{d^3y}{dx^3} + ye^x = 0$are respectively - **3, 1**

Miscellaneous Exercise 8 | Q 1.02 | Page 171

Choose the correct alternative.

The order and degree of $\left[1 + \left(\frac{dy}{dx}\right)^3\right]^{\frac{2}{3}} = 8\frac{d^3y}{dx^3}$ are respectively.

1. 3, 1
2. 1, 3
3. **3, 3**
4. 1, 1

Solution:

The order and degree of $\left[1 + \left(\frac{dy}{dx}\right)^3\right]^{\frac{2}{3}} = 8\frac{d^3y}{dx^3}$ are respectively - **3, 3**

Explanation

$$\left[1 + \left(\frac{dy}{dx}\right)^3\right]^{\frac{2}{3}} = 8\frac{d^3y}{dx^3}$$

Taking cube on both sides, we get

$$\left[1 + \left(\frac{dy}{dx}\right)^3\right]^{\frac{2}{3}} = 8^3\left(\frac{d^3y}{dx^3}\right)^3$$

∴ By definition of order and degree,

Order : 3; Degree : 3

Miscellaneous Exercise 8 | Q 1.03 | Page 171

Choose the correct alternative.

The differential equation of y = $k_1 + \frac{k_2}{x}$ is

Options

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = 0$$

$$x\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = 0$$

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} = 0$$

$$x\frac{d^2y}{dx^2} - 2\frac{dy}{dx} = 0$$

Solution:

The differential equation of $y = k_1 + \frac{k_2}{x}$ is $x\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = 0$

Explanation

$$y = k_1 + \frac{k_2}{x}$$

$\therefore$ xy = xk_1 + k_2

Differentiating w.r.t. x, we get

$$y + x\frac{dy}{dx} = k_1$$

Again, differentiating w.r.t. x, we get

$$\frac{dy}{dx} + \frac{dy}{dx} + x\frac{d^2y}{dx^2} = 0$$

$$\therefore x\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = 0$$

Miscellaneous Exercise 8 | Q 1.04 | Page 171

Choose the correct alternative.

The differential equation of $y = k_1e^x + k_2e^{-x}$ is

Options

$$\frac{d^2y}{dx^2} - y = 0$$

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} = 0$$

$$\frac{d^2y}{dx^2} + y\frac{dy}{dx} = 0^*$$

$$\frac{d^2y}{dx^2} + y = 0$$

Solution:

The differential equation of $y = k_1e^x + k_2e^{-x}$ is $\frac{d^2y}{dx^2} - y = 0$

Explanation

$$y = k_1e^x + k_2e^{-x}$$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = k_1 e^x - k_2 e^{-x}$$

Again, differentiating w.r.t. x, we get

$$\frac{d^2y}{dx^2} = k_1 e^x + k_2 e^{-x}$$

$$\therefore \frac{d^2y}{dx^2} = y$$

$$\therefore \frac{d^2y}{dx^2} - y = 0$$

Miscellaneous Exercise 8 | Q 1.05 | Page 171

Choose the correct alternative.

The solution of dy/dx = 1 is

1. x + y = c
2. xy = c
3. $x^2 + y^2 = c$
4. **y − x = c**

Solution:

The solution of $\frac{dy}{dx}$ = 1 is **y − x = c**

Explanation

$$\frac{dy}{dx} = 1$$

∴ dy = dx

Integrating on both sides, we get

$$\int 1dy = \int 1dx$$

$\therefore$ y = x + c

$\therefore$ y - x = c

Miscellaneous Exercise 8 | Q 1.06 | Page 171

Choose the correct alternative.

The solution of $\frac{dy}{dx} + \frac{x^2}{y^2} = 0$ is

1. $x^3 + y^3 = 7$
2. $x^2 + y^2 = c$
3. **$x^3 + y^3 = c$**
4. x + y = c

Solution:

The solution of $\frac{dy}{dx} + \frac{x^2}{y^2} = 0$ is **$x^3 + y^3 = c$**

Miscellaneous Exercise 8 | Q 1.07 | Page 172

Choose the correct alternative.

The solution of $x\frac{dy}{dx} = y$ log y is

1. $y = ae^x$
2. $y = be^{2x}$
3. $y = be^{-2x}$
4. **$y = e^{ax}$**

Solution:

The solution of $x\frac{dy}{dx} = y$ log y is **$y = e^{ax}$**

$$x\frac{dy}{dx} = y \log y$$

$$\therefore \frac{dy}{y\log y} = \frac{dx}{x}$$

Integrating on both sides, we get

$$\int \frac{dy}{y\log y} = \int \frac{1}{x}dx$$

$\therefore$ log log(y)= log x + log a

$\therefore$ log log(y)= log xa

$\therefore$ log(y)= ax

$\therefore$ y = e^{ax}

Miscellaneous Exercise 8 | Q 1.08 | Page 172

Choose the correct alternative.

Bacteria increases at the rate proportional to the number present. If the original number M doubles in 3 hours, then the number of bacteria will be 4M in

1. 4 hours
2. **6 hours**
3. 8 hours
4. 10 hours

Solution: Bacteria increases at the rate proportional to the number present. If the original number M doubles in 3 hours, then the number of bacteria will be 4M in **6 hours**

Miscellaneous Exercise 8 | Q 1.09 | Page 172

Choose the correct alternative.

The integrating factor of $\frac{dy}{dx} + y = e^{-x}$

1. x

2. – x
3. **e**
4. e^{-x}

Solution:

The integrating factor of $\frac{dy}{dx} + y = e^{-x}$ - **e**

Explanation

$$\frac{dy}{dx} + y = e^{-x}$$

The given equation is of the form $\frac{dy}{dx} + py = Q$

where, P = 1 and Q = e^{-x}

∴ I.F. = $e^{\int pdx} = e^{\int 1dx} = e^{x}$

Miscellaneous Exercise 8 | Q 1.1 | Page 172

Choose the correct alternative.

The integrating factor of $\frac{dy}{dx} - y = e^{x}$ is e^{x}, then its solution is

1. **$ye^{-x} = x + c$**
2. $ye^{x} = x + c$
3. $ye^{x} = 2x + c$
4. $ye^{-x} = 2x + c$

Solution:

The integrating factor of $\frac{dy}{dx} - y = e^{x}$ is e^{x}, then its solution is **$ye^{-x} = x + c$**

Explanation

$$\frac{dy}{dx} - y = e^x$$

Here, I.F. = e^{-x} , Q = e^x

∴ Solution of the given equation is

$$y(I.F.) = \int Q(I.F.)dx + c$$

$$\therefore ye^{-x} = \int e^x e^{-x} dx + c$$

$$\therefore ye^{-x} = \int 1dx + c$$

∴ $ye^{-x} = x + c$

Miscellaneous Exercise 8 | Q 2.1 | Page 172

Fill in the blank:

The order of highest derivative occurring in the differential equation is called ___________ of the differential equation.

Solution: The order of highest derivative occurring in the differential equation is called **order** of the differential equation.

Miscellaneous Exercise 8 | Q 2.2 | Page 172

Fill in the blank:

The power of the highest ordered derivative when all the derivatives are made free from negative and / or fractional indices if any is called __________ of the differential equation.

Solution: The power of the highest ordered derivative when all the derivatives are made free from negative and/or fractional indices if any is called **degree** of the differential equation.

Miscellaneous Exercise 8 | Q 2.3 | Page 172

Fill in the blank:

A solution of differential equation which can be obtained from the general solution by giving particular values to the arbitrary constants is called ___________ solution.

Solution: A solution of differential equation which can be obtained from the general solution by giving particular values to the arbitrary constants is called **particular** solution.

Miscellaneous Exercise 8 | Q 2.4 | Page 172

Fill in the blank:

Order and degree of a differential equation are always __________ integers.

Solution: Order and degree of a differential equation are always **positive** integers.

Miscellaneous Exercise 8 | Q 2.5 | Page 172

Fill in the blank:

The integrating factor of the differential equation dy/dx – y = x is __________

Solution:

The integrating factor of the differential equation $\frac{dy}{dx} - y = x$ is **e^{-x}**

Miscellaneous Exercise 8 | Q 2.6 | Page 172

Fill in the blank:

The differential equation by eliminating arbitrary constants from bx + ay = ab is __________.

Solution:

The differential equation by eliminating arbitrary constants from bx + ay = ab is $\frac{d^2y}{dx^2} = 0$

Explanation

bx + ay = ab

Differentiating w.r.t. x, we get

$$b + a\frac{dy}{dx} = 0$$

$$\therefore \frac{dy}{dx} = \frac{-b}{a}$$

Again, differentiating w.r.t. x, we get

$$\frac{d^2y}{dx^2} = 0$$

Miscellaneous Exercise 8 | Q 3.1 | Page 172

State whether the following is True or False:

The integrating factor of the differential equation $\frac{dy}{dx} - y = x$ is e^{-x}

1. **True**
2. False

Solution:

The integrating factor of the differential equation $\frac{dy}{dx} - y = x$ is e^{-x} - **True**

Miscellaneous Exercise 8 | Q 3.2 | Page 172

State whether the following is True or False:

Order and degree of a differential equation are always positive integers.

1. **True**
2. False

Solution: Order and degree of a differential equation are always positive integers.- **True**

Miscellaneous Exercise 8 | Q 3.3 | Page 172

State whether the following is True or False:

The degree of a differential equation is the power of the highest ordered derivative when all the derivatives are made free from negative and/or fractional indices if any.

1. **True**
2. False

Solution: The degree of a differential equation is the power of the highest ordered derivative when all the derivatives are made free from negative and/or fractional indices if any. - **True**

Miscellaneous Exercise 8 | Q 3.4 | Page 172

State whether the following is True or False:

The order of highest derivative occurring in the differential equation is called degree of the differential

1. True
2. **False**

Solution: The order of highest derivative occurring in the differential equation is called degree of the differential equation. - **False**

Miscellaneous Exercise 8 | Q 3.5 | Page 172

State whether the following is True or False:

The power of the highest ordered derivative when all the derivatives are made free from negative and / or fractional indices if any is called order of the differential equation.

1. True
2. **False**

Solution: The power of the highest ordered derivative when all the derivatives are made free from negative and / or fractional indices if any is called order of the differential equation. - **False**

Miscellaneous Exercise 8 | Q 3.6 | Page 172

State whether the following is True or False:

The degree of the differential equation $e^{\frac{dy}{dx}} = \frac{dy}{dx} + c$ is not defined.

1. **True**
2. False

Solution:

The degree of the differential equation $e^{\frac{dy}{dx}} = \frac{dy}{dx} + c$ is not defined. - **True**

Miscellaneous Exercise 8 | Q 4.01 | Page 172

Find the order and degree of the following differential equation:

$$\left[\frac{d^3y}{dx^3} + x\right]^{\frac{3}{2}} = \frac{d^2y}{dx^2}$$

Solution:

$$\left[d^3\frac{y}{dx^3} + x\right]^{\frac{3}{2}} = \frac{d^2y}{dx^2}$$

Squaring on both sides, we get

$$\left[\frac{d^3y}{dx^3} + x\right]^3 = \left(\frac{d^2y}{dx^2}\right)^2$$

By definition of order and degree,

Order : 3 ; Degree : 3

Miscellaneous Exercise 8 | Q 4.01 | Page 172

Find the order and degree of the following differential equation:

$$x+\frac{dy}{dx}=1+\left(\frac{dy}{dx}\right)^2$$

Solution:

$$x+\frac{dy}{dx}=1+\left(\frac{dy}{dx}\right)^2$$

By definition of order and degree,

Order : 1 ; Degree : 2

Miscellaneous Exercise 8 | Q 4.02 | Page 172

Verify y = log x + c is a solution of the differential equation

$$x\frac{d^2y}{dx^2}+\frac{dy}{dx}=0$$

Solution:

y = log x + c

Differentiating w.r.t. x, we get

$$\frac{dy}{dx}=\frac{1}{x}$$

$$\therefore x\frac{dy}{dx}=1$$

Again, differentiating w.r.t. x, we get

$$x\frac{d^2y}{dx^2}+\frac{dy}{dx}=0$$

$\therefore$ Given function is a solution of the given differential equation.

Miscellaneous Exercise 8 | Q 4.03 | Page 172

Solve the differential equation:

$$\frac{dy}{dx} = 1 + x + y + xy$$

Solution:

$$\frac{dy}{dx} = 1 + x + y + xy$$

=(1 + x)+ y (1+x)

= (1 + x) (1 + y)

$$\therefore \frac{dy}{1+y} = (1+x)dx$$

Integrating on both sides, we get

$$\int \frac{dy}{1+y} = \int (1+x)dx$$

$$\therefore \log|1+y| = x + \frac{x^2}{2} + c$$

Miscellaneous Exercise 8 | Q 4.03 | Page 172

Solve the differential equation:

$$e^{\frac{dy}{dx}} = x$$

Solution:

$$e^{\frac{dy}{dx}} = x$$

$$\therefore \frac{dy}{dx} = \log x$$

∴ dy = log x dx

Integrating on both sides, we get

$$\int dy = \int (\log x) 1 dx$$

$$\therefore y = \log x \int 1 dx - \int \left[\frac{d}{dx}(\log x) \int 1 dx\right] dx$$

$$= x \log x - \int \frac{1}{x} . x dx$$

$$= x \log x - \int dx$$

∴ y = x log x - x + c

Miscellaneous Exercise 8 | Q 4.03 | Page 173

Solve the differential equation:

dr = a r dθ − θ dr

Solution:

dr = a r dθ – θ dr

∴ (1 + θ) dr = a r dθ

$$\therefore \frac{dr}{r} = a \frac{d\theta}{(1+\theta)}$$

Integrating on both sides, we get

$$\int \frac{dr}{r} = a \int \frac{d\theta}{1+\theta}$$

log | r | = a log | 1 + θ| + log | c |

∴ log | r | = log | c(1 + θ)^a|

∴ r = c (1 + θ)^a

Miscellaneous Exercise 8 | Q 4.03 | Page 173

Solve the differential equation:

Find the differential equation of family of curves y = e^x (ax + bx^2), where A and B are arbitrary constants.

Solution: y = e^x (ax + bx^2)

∴ y = xe^x (a + bx)

$$\therefore \frac{y}{xe^x} = a + bx$$

Differentiating w.r.t. x, we get

$$\frac{xe^x \frac{dy}{dx} - (y(e^x + xe^x))}{x^2 (e^x)^2} = b$$

$$\therefore \frac{x\frac{dy}{dx} - y - xy}{x^2 e^x} = b$$

Again, differentiating w.r.t. x, we get

$$\frac{\cdot x^2 e^x \left(\frac{dy}{dx} + x\frac{d^2y}{dx^2} - \frac{dy}{dx} - y - x\frac{dy}{dx}\right) - \left(x\frac{dy}{dx} - y - xy\right)\left(x^2 e^x + 2xe^x\right)}{\left(x^2 e^x\right)^2} = 0$$

$$\therefore xe^x \left[x\left(x\frac{d^2y}{dx^2} - y - x\frac{dy}{dx}\right) - \left(x\frac{dy}{dx} - y - xy\right)(x+2)\right] = 0$$

$$\therefore x^2\frac{d^2y}{dx^2} - xy - x^2\frac{dy}{dx} - \left(x^2\frac{dy}{dx} - xy - x^2y + 2x\frac{dy}{dx} - 2y - 2xy\right) = 0$$

$$\therefore x^2\frac{d^2y}{dx^2} - xy - x^2\frac{dy}{dx} - x^2\frac{dy}{dx} + xy + x^2y - 2x\frac{dy}{dx} + 2y + 2xy = 0$$

$$\therefore x^2\frac{d^2y}{dx^2} - 2x^2\frac{dy}{dx} - 2x\frac{dy}{dx} + x^2y + 2y + 2xy = 0$$

Miscellaneous Exercise 8 | Q 4.04 | Page 173

Solve

$$\frac{dy}{dx} = \frac{x+y+1}{x+y-1} \ when \ x = \frac{2}{3} \text{ and } y = \frac{1}{3}$$

Solution:

$$\frac{dy}{dx} = \frac{x+y+1}{x+y-1} \quad \text{...(i)}$$

Put x + y = t ...(ii)

∴ y = t - x

Differentiating w.r.t. x, we get

$$\therefore \frac{dy}{dx} = \frac{dt}{dx} - 1 \text{ ...(iii)}$$

Substituting (ii) and (iii) in (i), we get

$$\frac{dt}{dx} - 1 = \frac{t+1}{t-1}$$

$$\therefore \frac{dt}{dx} = \frac{t+1}{t-1} + 1 = \frac{t+1+t-1}{t-1}$$

$$\therefore \frac{dt}{dx} = \frac{2t}{t-1}$$

$$\therefore \left(\frac{t-1}{t}\right)dt = 2dx$$

$$\therefore \left(1-\frac{1}{t}\right)dt = 2dx$$

Integrating on both sides, we get

$$\int\left(1-\frac{1}{t}\right)dt = 2\int dx$$

∴ t - log | t | = 2x + c

∴ x + y -log |x + y| = 2x + c

∴ - log |x + y| = x - y + c

Putting $x = \dfrac{2}{3}$ and $y = \dfrac{1}{3}$, we get

$$-\log(1) = \frac{1}{3} + c$$

∴ c = $\dfrac{1}{3}$

$$\therefore -\log|x+y| = x - y\frac{1}{3}$$

$$\therefore \log|x+y| = y - x + \frac{1}{3}$$

Miscellaneous Exercise 8 | Q 4.05 | Page 173

Solve

y dx – x dy = −log x dx

Solution: y dx – x dy = – log x dx

Dividing throughout by dx, we get

$$y - x\frac{dy}{dx} = -\log x$$

$$\therefore -x\frac{dy}{dx} + y = -\log x$$

$$\therefore \frac{dy}{dx} - \frac{1}{x}y = \frac{\log x}{x}$$

The given equation is of the form

$$\frac{dy}{dx} + py = Q$$

where, $P = -\frac{1}{x}$ and $Q = \frac{\log x}{x}$

∴ I.F. = $e^{\int pdx = e^{\int -\frac{1}{x}dx} e^{-\log x}}$

= $e^{\log x^{-1}} = x^{-1} = \frac{1}{x}$

∴ Solution of the given equation is

$$y(I.F.) = \int Q(I.F.)dx + c$$

$$\therefore \frac{y}{x} = \int \frac{\log x}{x} \times \frac{1}{x}dx + c$$

In R. H. S., put log x = t ...(i)

∴ x = e^t

Differentiating (i) w.r.t. x, we get

$$\frac{1}{x}dx = dt$$

$$\therefore \frac{y}{x} = \int \frac{t}{e^t} dt + c$$

$$\therefore \frac{y}{x} = \int te^t \, dt + c$$

$$= t \int e^{-t} dt - \int \left(\frac{d}{dt}(t) \times \int e^{-t} dt \right) dt + c$$

$$= -te^{-t} - \int (-e^{-t}) dt + c$$

$$= -te^{-t} + \int e^{-t} dt + c$$

$$= - te^{-t} - e^{-t} + c$$

$$= \frac{-t - t}{e^t} + c$$

$$= \frac{-\log x - 1}{x} + c$$

$\therefore$ y = cx – (1 + log x)

$\therefore$ log x + y + 1 = cx

Miscellaneous Exercise 8 | Q 4.06 | Page 173

Solve

$$y \log y \frac{dy}{dx} + x - \log y = 0$$

Solution:

$$y \log y \frac{dy}{dx} + x - \log y = 0$$

$$\therefore \frac{dx}{dy} + \frac{1}{y \log y} x = \frac{1}{y}$$

The given equation is of the form $\frac{dx}{dy} + px = Q$

where, $P = \frac{1}{y \log y}$ and $Q = \frac{1}{y}$

$\therefore I.F. = e^{\int^{p} dy} = e^{\int \frac{1}{y \log y} dy} = e^{\log|\log y|} = \log y$

$\therefore$ Solution of the given equation is

$$x(I.F.) = \int Q(I.F.)dy + c_1$$

$$\therefore x.\log y = \int \frac{1}{y} \log y \, dy + c_1$$

In R. H. S., put log y = t

Differentiating w.r.t. x, we get

$$\frac{1}{y} dy = dt$$

$$\therefore x \log y = \int t dt + c_1 = \frac{t^2}{2} + c_1$$

In R. H. S., put log y = t

Differentiating w.r.t. x, we get

$$\frac{1}{y} dy = dt$$

$$\therefore x \log y = \int t dt + c_1 = \frac{t^2}{2} + c_1$$

$$\therefore x \log y = \frac{(\log y)^2}{2} + c_1$$

$\therefore$ 2x log y = $(\log y)^2$ + c …[$2c_1 = c$]

Miscellaneous Exercise 8 | Q 4.07 | Page 173

Solve:

(x + y) dy = a^2 dx

Solution:

(x + y) dy = a^2 dx

$\therefore \frac{dy}{dx} = \frac{a^2}{x+y}$ ...(i)

Put x + y = t ...(ii)

$\therefore$ y = t - x

Differentiating w.r.t. x, we get

$\therefore \frac{dy}{dx} = \frac{dt}{dx} - 1$ ....(iii)

Substituting (ii) and (iii) in (i), we get

$\frac{dt}{dx} - 1 = \frac{a^2}{t}$

$\therefore \frac{dt}{dx} = \frac{a^2}{t} + 1$

$\therefore \frac{dt}{dx} = \frac{a^2 + t}{t}$

$\therefore \frac{t}{a^2 + t} dt = dx$

Integrating on both sides, we get

$$\int \frac{(a^2+t)-a^2}{a^2+t}\,dt = \int dx$$

$$\therefore \int 1dt - a^2 \int \frac{1}{a^2+t}dt = \int dx$$

$\therefore$ t - a^2 log $|a^2$ + t| = x + c_1

$\therefore$ x + y - a^2 log $|a^2$ + x + y| = x + c_1

$\therefore$ y - a^2 log $|a^2$ + x + y| = c_1

$\therefore$ y - c_1 = a^2 log $|a^2$ + x + y|

$$\therefore \frac{y}{a^2} - \frac{c_1}{a^2} = \log\left|a^2+x+y\right|$$

$$\therefore a^2 + x + y = e^{a^{\frac{y}{2}}}.e^{a^{\frac{-c1}{2}}}$$

$$\therefore a^2 + x + y = ce^{a^{\frac{y}{2}}} \ldots \left[c = e^{a^{\frac{-c1}{2}}}\right]$$

Miscellaneous Exercise 8 | Q 4.08 | Page 173

Solve

$$\frac{dy}{dx} + \frac{2}{x}y = x^2$$

Solution:

$$\frac{dy}{dx} + \frac{2}{x}y = x^2$$

The given equation is of the form

$$\frac{dy}{dx} + py = Q$$

$$where, P = \frac{2}{x} \text{ and } Q = x^2$$

$\therefore$ I.F. $= e^{\int pdx = e^{2\int \frac{1}{x}dx} e = 2\log x = e^{\log x^2} = x^2}$

$\therefore$ Solution of the given equation is

$$y(I.F.) = \int Q(I.F.)dx + c_1$$

$$y(x^2) = \int x^2 \times x^2 dx + c_1$$

$$\therefore x^2 y = x^4 \int dx + c_1$$

$$\therefore x^2 y = \frac{x^5}{5} + c_1$$

$\therefore 5x^2 y = x^5 + c$ ...[$c = 5c_1$]

Miscellaneous Exercise 8 | Q 4.09 | Page 173

The rate of growth of population is proportional to the number present. If the population doubled in the last 25 years and the present population is 1 lac, when will the city have population 4,00,000?

Solution:

Let ‘x’ be the population at time ‘t’ years.

$$\therefore \frac{dx}{dt} \propto x$$

$\therefore \frac{dx}{dt} = kx$, where k is the constant of proportionality.

Integrating on both sides, we get

$$\int \frac{dx}{x} = k \int dt$$

∴ log x = kt + c ...(i)

When t = 0, x = 50000

∴ log (50000) = k (0) + c

∴ c = log (50000)

∴ log x = kt + log (50000) ...(ii)[From (i)]

When t = 25, x = 100000, we have

log (100000) = 25k + log (50000)

∴ log 2 = 25k

∴ $k = \frac{1}{25} \log 2$ ...(iii)

When x = 400000, we get

$\log(400000) = \left[\frac{1}{25}\log(2)\right] t + \log(50000)$ ...[From (ii) and (iii)]

$$\therefore \log\left[\frac{400000}{50000}\right] = \frac{t}{25} \log 2$$

$$\therefore \log 8 = \frac{t}{25} \log 2$$

$$\therefore 3 \log 2 = \frac{t}{25} \log 2$$

$$\therefore \frac{t}{25} = 3$$

$\therefore$ t = 75 years.

Thus, the population will be 4,00,000 after 75 – 25 = 50 years from present date.

Miscellaneous Exercise 8 | Q 4.1 | Page 173

The resale value of a machine decreases over a 10 year period at a rate that depends on the age of the machine. When the machine is x years old, the rate at which its value is changing is ₹ 2200 (x − 10) per year. Express the value of the machine as a function of its age and initial value. If the machine was originally worth ₹1,20,000, how much will it be worth when it is 10 years old?

Solution: Let 'y' be the value of the machine when machine is 'x' years old.

$\therefore$ According to the given condition,

$$\frac{dy}{dx} = 2200(x - 10)$$

$\therefore dy = 2200(x - 10)\, dx$

Integrating on both sides, we get

$\int 1\, dy = 2200 \int (x-10)\, dx$

$$\therefore y = 2200\left(\frac{x^2}{2} - 10x\right) + c$$

$\therefore y = 1100 x^2 - 22{,}000 x + c$

when x = 0, y = 1,20,000

$\therefore 1{,}20{,}000 = 1100(0)^2 - 22{,}00(0) + c$

$\therefore$ c = 1,20,000

$\therefore$ The value of the machine can be expressed as a function of it's age as

y = $1{,}100x^2 - 22{,}000x + 1{,}20{,}000$

Initial value: when x = 0, y = 1,20,000

$\therefore$ when x = 10,

y = $1100(10)^2 - 22{,}000(10) + 1{,}20{,}000$

= 10,000

$\therefore$ The machine will worth ₹ 10,000 when it is 10 years old.

Miscellaneous Exercise 8 | Q 4.11 | Page 173

y2 dx + $(xy + x^2)$dy = 0

Solution:

y2 dx + $(xy + x^2)$dy = 0

$\therefore (xy + x^2)\ dy = -y^2\ dx$

$$\therefore \frac{dy}{dx} = -\frac{y^2}{xy + x^2} \quad \text{...(i)}$$

Put y = tx ...(ii)

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = t + x\frac{dt}{dx} \quad \text{...(iii)}$$

Substituting (ii) and (iii) in (i), we get

$$\therefore t + x\frac{dt}{dx} = \frac{-t^2x^2}{x.tx + x^2}$$

$$\therefore t+x\frac{dt}{dx}=\frac{-t^2x^2}{x^2(t+1)}$$

$$\therefore x\frac{dt}{dx}=\frac{-t^2}{t+1}-t$$

$$\therefore\ x\frac{dt}{dx}=\frac{-t^2-t^2-t}{t+1}$$

$$\therefore\ x\frac{dt}{dx}=\frac{-\left(2t^2+t\right)}{t+1}$$

$$\therefore \frac{t+1}{2t^2+t}dt=-\frac{1}{x}dx$$

Integrating on both sides, we get

$$\int\frac{t+1}{2t^2+t}dt=-\int\frac{1}{x}dx$$

$$\therefore\int\frac{2t+1-t}{t(2t+1)}dt=-\int\frac{1}{x}dx$$

$$\therefore\int\frac{1}{t}dt-\int\frac{1}{2t+1}dt=-\int\frac{1}{x}dx$$

$$\therefore \log|t|-\frac{1}{2}\log|2t+1|=-\log|x|+\log|c|$$

$\therefore$ 2log| t | -log |2t + 1| = -2log |x| + 2 log |c|

$$\therefore 2\log\left|\frac{y}{x}\right|-\log\left|\frac{2y}{x}+1\right|=-2\log|x|+2\log|c|$$

$\therefore$ 2log |y| - 2log |x| - log |2y + x| + log |x| = - 2log |x| + 2log |c|

$\therefore$ log $|y^2|$ + log |x| = log $|c^2|$+ log |2y + x|

$\therefore$ log $|y^2x|$ = log $|c^2(x + 2y)|$

$\therefore \ \log|y^2x| = \log|c^2(x + 2y)|$

$\therefore \ xy^2 = c^2 (x + 2y)$

Miscellaneous Exercise 8 | Q 4.12 | Page 173

$x^2y\ dx - (x^3 + y^3)\ dy = 0$

Solution: $x^2y\ dx - (x^3 + y^3)\ dy = 0$

$\therefore \ x^2y\ dx - (x^3 + y^3) = dy$

$$\therefore \ \frac{dy}{dx} = \frac{x^2y}{x^3 + y^3} \quad \text{...(i)}$$

Put y = tx ...(ii)

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = t + x\frac{dt}{dx} \quad \text{...(iii)}$$

Substituting (ii) and (iii) in (i), we get

$$t + x\frac{dt}{dx} = \frac{x^2.tx}{x^3 + t^3x^3}$$

$$\therefore t + x\frac{dt}{dx} = \frac{x^3.t}{x^3(1 + t^3)}$$

$$\therefore x\frac{dt}{dx} = \frac{t}{1 + t^3} - t$$

$$\therefore x\frac{dt}{dx} = \frac{-t^4}{1 + t^3}$$

$$\therefore \frac{1 + t^3}{t^4}dt = -\frac{dx}{x}$$

Integrating on both sides, we get

$$\int \frac{1+t^3}{t^4} dt = -\int \frac{1}{x} dx$$

$$\therefore \int \left(\frac{1}{t^4} + \frac{1}{t} \right) dt = -\int \frac{1}{x} dx$$

$$\therefore \int t^{-4} dt + \int \frac{1}{t} dt = -\int \frac{1}{x} dx$$

$$\therefore \frac{t^3}{-3} + \log|t| = -\log|x| + c$$

$$\therefore -\frac{1}{3t^3} + \log|t| = -\log|x| + c$$

$$\therefore -\frac{1}{3} \cdot \frac{1}{\left(\frac{y}{x}\right)^3} + \log\left|\frac{y}{x}\right| = -\log|x| + c$$

$$\therefore \frac{x^3}{3y^3} + \log|y| - \log|x| = -\log|x| + c$$

$$\therefore \log|y| - \frac{x^3}{3y^3} = c$$

Miscellaneous Exercise 8 | Q 4.13 | Page 173

$$xy\frac{dy}{dx} = x^2 + 2y^2$$

Solution:

$$xy\,\frac{dy}{dx} = x^2 + 2y^2$$

$$\therefore \frac{dy}{dx} = x^2 + \frac{2y^2}{xy} \quad \text{...(i)}$$

Put y = tx ...(ii)

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = t + x\frac{dt}{dx} \quad ...(iii)$$

Substituting (ii) and (iii) in (i), we get

$$t + x\frac{dt}{dx} = \frac{x^2 + 2t^2x^2}{x(tx)}$$

$$\therefore t + x\frac{dt}{dx} = \frac{x^2\left(1 + 2t^2\right)}{x^2t}$$

$$\therefore x\frac{dt}{dx}\frac{1 + 2t^2}{t} - t = \frac{1 + t^2}{t}$$

$$\therefore \frac{t}{1 + t^2}dt = \frac{1}{x}dx$$

Integrating on both sides, we get

$$\frac{1}{2}\int \frac{2t}{1 + t^2}\, dt = \int \frac{dx}{x}$$

$$\therefore \frac{1}{2}\log\left|1 + t^2\right| = \log|x| + \log|c|$$

$\therefore$ log $|1 + t^2|$ = 2 log $|x|$ + 2log $|c|$

= log $|x^2|$ + log $|c^2|$

$\therefore$ log $|1 + t^2|$ = log $|c^2 x^2|$

$\therefore 1 + t^2 = c^2x^2$

$$\therefore 1 + \frac{y^2}{x^2} = c^2x^2$$

$\therefore x^2 + y^2 = c^2 x^4$

Miscellaneous Exercise 8 | Q 4.14 | Page 173

$$\left(x+2y^3\right)\frac{dy}{dx}=y$$

Solution:

$$\left(x+2y^3\right)\frac{dy}{dx}=y$$

$$\therefore \frac{x}{y}+2y^2=\frac{dx}{dy}$$

$$\therefore \frac{dx}{dy}-\frac{1}{y}x=2y^2$$

The given equation is of the form

$$\frac{dx}{dy}+px=Q$$

where, $P=-\frac{1}{y}$ and $Q=2y^2$

$\therefore$ I.F. $=e^{\int pdy}=e^{-\int \frac{1}{y}dy}$

$=e^{-\log y}=e^{\log y-1}$

= `y^ -1 = 1/ y`

$\therefore$ Solution of the given equation is

x(I.F.) = ∫Q(I.F.) dy + c

$$\therefore \frac{x}{y}2\int\frac{y^2}{y}dy+c$$

$$\therefore \frac{x}{y}2\int ydy+c$$

$$\therefore \frac{x}{y}2\int \frac{y^2}{y}+c$$

$\therefore$ x = y(c + y^2)

Miscellaneous Exercise 8 | Q 4.15 | Page 173

y dx – x dy + log x dx = 0

Solution: y dx – x dy + log x dx = 0

y dx – x dy = - log x dx

Dividing throughout by dx, we get

$$y-x\frac{dy}{dx}=-\log x$$

$$\therefore -x\frac{dy}{dx}+y=-\log x$$

$$\therefore \frac{dy}{dx}-\frac{1}{xy}=\frac{\log x}{x}$$

The given equation is of the form

$$\frac{dy}{dx}+py=Q$$

where, $P=-\frac{1}{x}$ and $Q=\frac{\log x}{x}$

$\therefore$ I.F. = $e^{\int pdx}=e^{\int -\frac{1}{x}dx}e^{-\log x}$

= $e^{\log x^{-1}}=x^{-1}=\frac{1}{x}$

∴ Solution of the given equation is

$$y(I.F.) = \int Q(I.F.)dx + c$$

$$\therefore \frac{y}{x} = \int \frac{\log x}{x} \times \frac{1}{x} dx + c$$

In R. H. S., put log x = t ...(i)

∴ x = e^t

Differentiating (i) w.r.t. x, we get

$$\frac{1}{x} dx = dt$$

$$\therefore \frac{y}{x} = \int \frac{t}{e^t} dt + c$$

$$\therefore \frac{y}{x} = \int te^t \, dt + c$$

$$= t \int e^{-t} dt - \int \left(\frac{d}{dt}(t) \times \int e^{-t} dt \right) dt + c$$

$$= -te^{-t} - \int (-e^{-t}) dt + c$$

$$= -te^{-t} + \int e^{-t} dt + c$$

$$= -te^{-t} - e^{-t} + c$$

$$= \frac{-t - t}{e^t} + c$$

$$= \frac{-\log x - 1}{x} + c$$

∴ y = cx – (1 + log x)

Miscellaneous Exercise 8 | Q 4.16 | Page 173

$$\frac{dy}{dx} = \log x$$

Solution:

$$\frac{dy}{dx} = \log x$$

∴ dy = log x dx

Integrating on both sides, we get

∫ 1 dy = ∫ (log x × 1) dx

$$\therefore y = \log x\left(\int 1dx\right) - \int\left[\frac{d}{dx}(\log x)\int 1dx\right]$$

$$\therefore y = \log x(x) - \int\left(\frac{1}{x} \times x\right)dx$$

= x log x – ∫ 1dx

∴ y = x log x – x + c

Miscellaneous Exercise 8 | Q 4.17 | Page 173

Solve

$$y \log y \frac{dx}{dy} = \log y - x$$

Solution:

$$y \log y \frac{dx}{dy} = \log y - x$$

$$y \log y \frac{dx}{dy} + x = \log y$$

$$\therefore \frac{dx}{dy} + \frac{1}{y\log y}x = \frac{1}{y}$$

The given equation is of the form $\frac{dx}{dy} + px = Q$

where, $P = \frac{1}{y\log y}$ and $Q = \frac{1}{y}$

$$\therefore I.F. = e^{\int pdy = e^{\int \frac{1}{y\log y} dy = e^{\log|\log y| = \log y}}}$$

∴ Solution of the given equation is

$$x(I.F.) = \int Q(I.F.)dy + c_1$$

$$\therefore x.\log y = 1y \int \log y \, dy + c_1$$

In R. H. S., put log y = t

Differentiating w.r.t. x, we get

$$\frac{1}{y}dy = dt$$

$$\therefore x\log y = tdt \int + c_1 = \frac{t^2}{2} + c_1$$

$$\therefore x\log y = \frac{(\log y)^2}{2} + c_1$$

∴ 2x log y = $(\log y)^2$ + c ...[$2c_1$ = c]

x log y = $\frac{1}{2}$ $(\log y)^2$ + c